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March 24, 1920: "Calculation of amplifying and detecting properties of electron tubes" by Gregory Breit Gr.
April 7: "Formulae for the approximation of factorials" by Flora D. Sutton Gr.
April 13: "Prime factors in the quadratic domain" by Professor Abraham Cohen.
April 21: "An analytical treatment of the quadrilateral" by Frank V. Morley Gr.
April 28: This meeting was devoted to the solution of problems which had been proposed at previous meetings for solution and to the proposal of new problems.
May 5: "Quaternions" by Professor Frank D. Murnaghan; "Wilkinson's theorem"[1] by Frank V. Morley Gr.
May 12: "Induction motors" by Charles T. Zahn Gr.
May 19: "A self-dual Lüroth quartic" by Professor Frank Morley.
May 26: "The parabolic group of line-to-line transformations" by Dr. Tobias Dantzig, Instructor in mathematics.

A very noticeable feature of the work of this club, as the record of its proceedings shows, has been the proposal and solution of problems. Some part of the time at nearly every meeting has been devoted to this kind of activity.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

THE RELATION OF CAUSTICS TO CERTAIN ENVELOPES.

By OTTO DUNKEL, Washington University.

A number of problems have been proposed in the MONTHLY relating to the envelope of circles moving in a prescribed fashion.[2] Many such problems may be solved more simply by pure geometry than by the usual calculus procedure and it is desired to indicate here a method which is applicable in such cases and also to show how important facts may be learned by making use of caustic curves and their properties (see *1920*, 225). Suppose that a variable circle rolls on a fixed curve Γ' while its center C remains on a second fixed curve Γ and that the envelope of the moving circle is to be studied. Let C and C' be the centers of two such circles touching Γ' in M and M', respectively, and let $Q'P'$ be their common chord. Draw the diameters $Q'CK$, $Q'C'K'$. As C' approaches C the chord CC' approaches the tangent at C to Γ, M' and Q' approach M, and K and K' approach the extremity N of the diameter MCN. Since the angles $\angle KP'Q'$,

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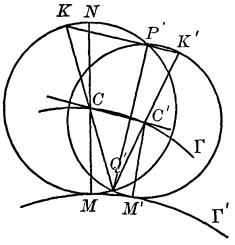
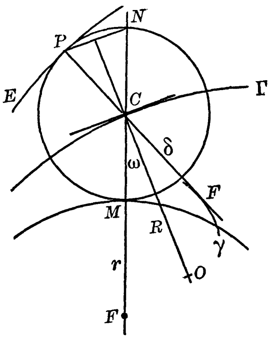
[1] Cf. MACKAY, *Proc. Edinb. Math. Soc.*, Vol. 11, 1893, p. 24; CASEY'S *Sequel to Euclid*, 6th ed., p. 66, exs. 30, 31.

[2] For example problems 2819 (*1920*, 134); 2827 (*1920*, 186); 2861 (*1920*, 428); 2868 (*1920*, 482).

$\angle K'P'Q'$ are right, the points K, K', P' lie in a straight line parallel to the chord CC' of Γ. Hence the limiting position of P', say P, is found by drawing through N a straight line parallel to the tangent at C to Γ. This line cuts the circle in P, its point of tangency with its envelope.[1]



Suppose now that any curve Γ is given with a fixed point F on its concave side and that γ is its caustic with respect to F. With F as a center and any radius r describe a circle, and let us suppose that a variable circle with its center C on Γ touches this circle at M, and that N is the other extremity of the diameter of the variable circle. Then the point P in which the variable circle touches its envelope$E$ is found as indicated above. Produce PC and it will be seen from the figure that this prolongation and FC are equally inclined to the tangent at C. Hence this prolongation touches the caustic γ in F'. Since γ is the envelope of the lines PF' it must be the evolute of the curve E, for PCF' is a normal at P. The relation between $FC = \delta$, $CF' = \delta'$, R the radius of curvature of Γ at C has been given in the article cited above and by use of this relation the radius of curvature $\rho = MC + \delta'$ of E may be calculated. Also if the equation of the envelope E is desired, it may be obtained from the construction given for P. In examples of envelopes to which this applies it will be found that these principles are more fruitful in results which lead to ready interpretation than the process of setting up equations and differentiating according to rule.



An interesting application of this method is as follows. Let s denote the length of arc of γ suitably measured from a fixed point on it. It is well known that $PF' + s = k$ (a constant). But $PF' = PC + \delta' = FC - r + \delta' = \delta + \delta' - r$. Hence

$$\delta + \delta' + s = r + k = \text{a constant}, \tag{1}$$

a known property of caustics which has been derived by other methods not quite so simple as the above. If F is at infinity, the circle whose center is F becomes a fixed straight line perpendicular to the incident parallel rays. If this line be taken as x-axis, then $MC = y$ for the point C and the above relation becomes

$$y + \delta' + s = \text{a constant}. \tag{2}$$

If the curve Γ is such that γ reduces to a point and hence $s = 0$, the equation (1) shows that Γ must be a central conic with foci F and F'. In the case of parallel rays, if γ is a point, (2) shows that Γ must be a parabola with F' as its focus.

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[1] In his solution of problem 2691 (*1919*, 131) R. A. Johnson gives a simpler derivation of an equivalent result.